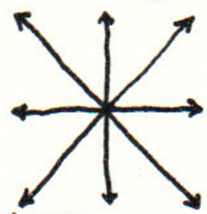
In the case n = 3 the situation is more complicated. For our purposes it suffices to investigate those lattices corresponding to imprimitive parallelohedrons on the boundary of Voronoi's fundamental domain. In the next section we will need to know the automorphisms of these lattices which leave the corresponding form Voronoi reduced. First let us look at the two dimensional lattices

case 1) a rectangular lattice



We check that there are 4 Minkowski reduced bases and 12 Voronoi reduced bases.

automorphisms:

2 $\pi/2$ rotations of order 4 which are given in terms of a Minkowski basis as $\pm$ S where S =

$$\pm \left(\begin{array}{l} C_1 \longrightarrow C_2 \\ C_2 \longrightarrow -C_1 \end{array} \right.$$

6 permutations of the normal coordinates. These automorphisms result from the symmetry of the form

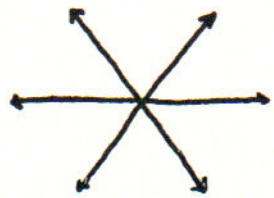
$$\varphi_0 = \tfrac{1}{2}\left[x_1^2 + x_2^2 + \ldots + x_n^2 + (x_1 + x_2 + \ldots + x_n)^2 \right]$$

which defines the fundamental domain. It can be shown that these permutations give all the automorphisms of the full fundamental domain. See Igusa [10].

In terms of a Voronoi basis these automorphisms are given as

$$\pm \left(\begin{array}{l} B_1 \longrightarrow B_2 \\ B_2 \longrightarrow B_1 \end{array} \right. \quad \pm \left(\begin{array}{l} B_1 \longrightarrow B_1 \\ B_2 \longrightarrow -B_1-B_2 \end{array} \right. \quad \pm \left(\begin{array}{l} B_1 \longrightarrow -B_1-B_2 \\ B_2 \longrightarrow B_1 \end{array} \right.$$

case 2) the hexagonal lattice



6 Minkowski reduced bases
12 Voronoi reduced bases

$2\pi/3$ rotations of order 6

in terms of a Minkowski basis

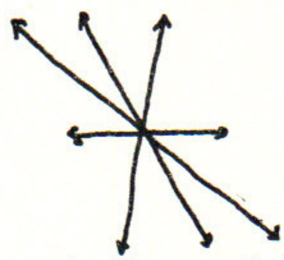
$$\pm\begin{cases} C_1 \longrightarrow C_2 \\ C_2 \longrightarrow -C_1 + C_2 \end{cases}$$

in terms of a Voronoi basis

$$\pm\begin{cases} B_1 \longrightarrow B_1 + B_2 \\ B_2 \longrightarrow -B_1 \end{cases}$$

6 permutations of the normal coordinates

case 3) general lattice in 2 dimensions



2 Minkowski reduced bases
8 Voronoi reduced bases

6 permutations of the normal coordinates.

In § 6 the 4 types of imprimitive parallelohedrons were expressed in terms of intersections of half-spaces. The bases of the lattices which give these equations are Voronoi reduced since the corresponding forms are. Besides the automorphisms corresponding to the 24 permutations of the normal coordinates, there are the following extra automorphisms:

type 1) right prism   6 $\pi/2$ rotations of order 4

type 2) hexagonal prism

$2\ \pi/3$ rotations of order 6
$4\ \pi/2$ rotations of order 4

type 3) rhombic dodecahedron

This case is more complicated and one needs to examine the half-space equations which define the body. Also, it is helpful to build a cardboard model. Such a model can be constructed from 12 equal rhombic faces which have face diagonals in the ratio $1/\sqrt{2}$. This follows from the fact that the body can be considered as made up of two equal cubes one of which is cut into 6 equal pyramids whose bases are placed on the faces of the other cube. The full automorphism group of the rhombic dodecahedron is of a large order. However, neglecting reflections and permutations of normal coordinates, there are

3 rotations of order 4 given in terms of a Voronoi basis as

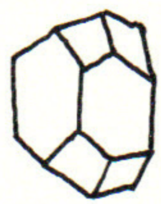
$$\left(\begin{matrix} B_1 & B_2 \\ B_2 & -B_1 \\ B_3 & B_1 + B_3 \end{matrix}\right. \quad \left(\begin{matrix} B_1 & -B_3 \\ B_2 & B_1+B_2+B_3 \\ B_3 & -B_2 \end{matrix}\right. \quad \left(\begin{matrix} B_1 & B_1 + B_3 \\ B_2 & B_2 + B_3 \\ B_3 & -B_1 - B_2 - B_3 \end{matrix}\right.$$

4 rotations of order 3

$$g\left(\begin{array}{l} B_1 \longrightarrow B_3 \\ B_2 \longrightarrow B_1+B_2+B_3 \\ B_3 \longrightarrow -B_1 - B_3 \end{array}\right) \qquad ef\left(\begin{array}{l} B_1 \longrightarrow B_1 + B_2 + B_3 \\ B_2 \longrightarrow B_3 \\ B_3 \longrightarrow -B_2 - B_3 \end{array}\right)$$

$$r\left(\begin{array}{l} B_1 \longrightarrow -B_1 - B_2 - B_3 \\ B_2 \longrightarrow -B_3 \\ B_3 \longrightarrow B_1 + B_3 \end{array}\right) \qquad s\left(\begin{array}{l} B_1 \longrightarrow -B_2 - B_3 \\ B_2 \longrightarrow B_1 + B_3 \\ B_3 \longrightarrow -B_1 \end{array}\right)$$

type 4) elongated dodecahedron

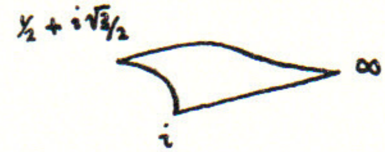


By examining the equations of the half-spaces we find that this body has 2 $\pi/4$ rotations of order 4 given in terms of a Voronoi basis as

$$\pm\left(\begin{array}{l} B_1 \longrightarrow B_2 \\ B_2 \longrightarrow -B_1 \\ B_3 \longrightarrow B_1 + B_3 \end{array}\right)$$

§9 The Siegel Space

We now mention an application of the preceding to the compactification of the quotient of the Siegel half-space $\mathfrak{S}_n$ by the full modular group Γ_n in the cases n = 2,3. By way of definition, $\mathfrak{S}_n = \mathcal{S}(n) + i\,\mathcal{P}(n)$ and $\Gamma_n = Sp(n,Z) = \left\{ M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \,\middle|\, {}^tM \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} M = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \text{ and } a,b,c,d \text{ are integral matrices} \right\}$. The action of $M \in \Gamma_n$ on $\gamma \in \mathfrak{S}_n$ is $M\gamma \rightarrow (a\gamma + b)(c\gamma + d)^{-1}$. Corresponding to the case n = 1 of the upper half-plane $\mathcal{H}/\Gamma_1 \cup \infty$, there are different types



of singularities on the boundary of the fundamental domain. First, there are singularities at points with finite stabilizer groups such as i and $\frac{1}{2} + i\sqrt{3}/2$. These singularities can be resolved in a finite covering. Secondly, there are the cusps or point equivalent to ∞. These singularities require a blowing-up process to be resolved. In particular, the fiber over the singular point will not be finite. For the cusps there are two methods of blowing-up available. The 1^{st} is due to Igusa [10] and uses the Fourier-Jacobi series corresponding to the part of the boundary in question. The second method is due to Satake [15] and uses a more geometric approach.

This section has to do with the singularities on the positive cone part of the half-space with finite stabilizer groups under the automorphisms of the cone. If we use the Voronoi fundamental domain for this action, then corresponding to the types of parallelohedrons on the boundary of the reduced domain there are different types of singularities on the compactification. These singularities may be resolved by a finite covering. In the previous section we calculated the automorphisms of the different parts of the boundary of the fundamental domain. Of these automorphisms, those corresponding to permutations of normal coordinates or to reflections are automorphisms of the whole domain. In the compactifications of Voronoi's fundamental domain these automorphisms identify different parts of the boundary. Hence, when we study the singularities of this compactified domain these automorphisms may be neglected. But, since the automorphisms of the positive cone being of the form $\begin{pmatrix} {}^tU & 0 \\ 0 & U^{-1} \end{pmatrix}\gamma = {}^tU\gamma U$

$\gamma \epsilon \mathcal{P}(n)$, $U \epsilon SL(n,Z)$ are also automorphisms of the full half-space $\mathfrak{G}_n$, in the compactification of the fundamental domain for Γ_n acting on $\mathfrak{G}_n$ the above mentioned automorphisms may also be neglected in the study of singularities on the boundary.

This, then, interprets the singularities arising from the automorphisms catalogued in the previous section.

APPENDIX

translated from Delone[4]

George Fedorevich Voronoi was born in 1868 on his father's country estate in Poltava county Russia. His father was the principal at the Nezhin Lyceum. Voronoi became interested in algebra at an early age. He went to Kiev to learn from Ermokov, the editor of a then current mathematical journal. There he studied quadratic irrationalities. Ermokov gave him the following ternary indefinite equation to solve in integers

$$x^2 + y^2 + z^2 = mxyz \quad m \text{ a positive integer.}$$

He couldn't solve it.

In 1885 he went to Petersburg University. In 1887 the money from his father ran out. He gave private lessons as a tutor for a pitable sum. In his diary he complains that this is tedious and exhausting and that his life in the student dormitory is not adapted well for studying. That is, he is disturbed by his collegues. Also, "Lectures in pure mathematics are the most fascinating. I prefer lectures on special cases of higher algebra to all the rest. The thing that bothers me chiefly is that I do not have enough talent."

In order to test himself he solves identities concerning symmetric functions, difficult definite integrals, and differential equations.

After taking upper division courses he wrote a senior thesis on a generalization of a theorem of Adams about Bernoulli numbers. Some time later Markov found this and published it for him. In 1889 he was retained as a graduate student and started work on his master's thesis, "Irrationalities of the Third Degree". He completed this and passed his oral examinations in 1894.

He then obtained an appointment at Warsaw. There he taught pure mathematics and worked on a generalization of the continued fraction algorithm which solves quadratic equations to those of the third degree. In May of 1897 he presented his doctoral dissertation on this at Petersburg. It was brillantly defended and the Russian Academy of Sciences crowned this dissertation with the Bunachowsky prize.

After this his creative life became increasingly intensive. However, he did not bother to publish his works. He wanted to wait for them to mature and repeatedly changed their presentation trying to find the most simple and elegant methods of presentation for his profound ideas. This culminated in the publishment of his two works in Crelle's Journal in 1908. These were the subject of his pride and the result of many years of stubborn thinking.

During this time, in 1904, Voronoi participated in the International Congress of Mathematicians in Heidelberg. There he met Minkowski who treated him with great interest, sympathy, and respect, for the themes of their work were closely related.

After the completion of the two aforementioned papers, and before the job of proofreading them was finished, Voronoi died. His sickness was an infection and stones in the gall bladder. This is the last entry in his diary, "I am acheiving great success in my research, but at the same time my health is constantly worsening. Yesterday, for the first time, I got a meaningful, clear idea about an algorithm which must solve all questions of the theory of forms under my research. And Yesterday I had a violent fit of gall bladder pains which prevented me from studying in the evening and sleeping at night. I am so afraid that the results of my long efforts which I acheived with such hard labor would perish together with me."

BIBLIOGRAPHY

[1] Alexandrov, A.D.: Konvex Polyeder. Berlin: Akademie-Verlag 1958.

[2] Baily, W.L. and Borel, A.:"Compactification of Arithmetric Quotients of Bounded Symmetric Domains". Annals of Math. 84(1966).

[3] Cassels, J.W.S.: An Introduction to the Geometry of Numbers. Berlin: Springer 1959.

[4] Delone, B.N.: The Petersburg School in the Theory of Numbers. Moscow: 1947(in russian).

[5] Dirichlet, L.: "Üeber die Reduction der Positiven Quadratischen Formen mit Drei Unbestimmter Ganzen Zahlen". Werke, Bd. II, S. 41.

[6] Fedorov: Elements de la Theorie des Figures. St. Petersburg: 1883(in russian).

[7] Gauss, C.F.: Werke,Bd. II, S. 188.

[8] Gottschling, E.: "Über die Fixpunktuntergruppen der Siegelschen Modulgruppe". Math. Annalen 143, 111-149, 399-430(1961).

[9] ________________ "Explizite Bestimmung der Randflache des Fundamental Bereiches der Modulgruppe Zweite Grades". Math. Annalen 138, 103-124(1959).

[10] Igusa, J.: "A Desingularization Problem in the Theory of Siegel Modular Functions". Math. Annalen 168, 228-260(1967).

[11] Lekkerkerker, C.G.:Geometry of Numbers. Groningen: Wolter-Noordhoff 1969.

[12] Minkowski, H.: "Dichteste Gitterformige Lagerung Kongruenten Korper". Werke, Bd. II S. 1.

[13] ____________ "Diskontinuitat Bereich Für Arithmetric Aquivalenz". Werke Bd. II, S. 53.

[14] Satake, I.: "Realization of Symmetric Domains as Siegel Domains of the Third Kind". Lecture Notes, U. of Calif., Berkeley, Fall 1972 (to appear).

[15] ____________ "The Arithmetric of Tube Domains". Bulletin of the American Math. Soc., Nov. 1973.

[16] Voronoi, G.F.: "Nouvelles Applications des Parametres Continus á la Théorie des Formes Quadratiques". Crelles Journal 133, 134, 136, 1907, 1908 (in three parts).

[17] Kelvin, Lord: <u>Baltimore Lectures on Molecular Dynamics</u>. London, 1904.