clear that c can be chosen among the common vertices of P_o and P_i.

<u>Theorem</u> Given P_o and the lattice made up of the centers of its translates, there exists uniquely a positive definite quadratic form such that P_o consists of those points x which satisfy the equation

$$\varphi(x) \leq \varphi(x-v) \quad \text{for all } v \text{ in the lattice.}$$

<u>Lemma 1</u> Let $H_{k,\ell}$ be the half-space which is bounded by the common face of P_k and P_ℓ and contains P_k. Then, the a_i can be so chosen that $H_{k,\ell}$ is determined by inequality

$$(a_\ell - a_k)\cdot(x - c) \geq 0 \qquad \text{with } a_o = 0 \text{ and } c$$

the common vertex of P_o, P_k, P_ℓ.

proof) Let P_o, P_1, ... , P_n surround the vertex c. Let E_m denote the common edge of those parallelohedra with the exception of P_m which surround c. The edge E_o passes through the interior of P_o. So, there must exist positive numbers t_1, ... , t_n such that the edge E_o satisfies the equations

$$t_1a_1(x - c) = t_2a_2(x - c) = \dots = t_na_n(x - c).$$

On the other hand, since the edge E_m belongs to the faces F_k, $k \neq m$, it satisfies the equations $a_k\cdot(x - c) = 0$, $k \neq m$. The hyperplane separating P_k and P_ℓ is determined by the edges E_m, $m \neq k,\ell$. By the foregoing, this hyperplane is determined by the equation

$t_k a_k (x - c) = t_\ell a_\ell \cdot (x - c)$, where if either k or ℓ is $= 0$ we put $t_0 = 1$, $a_0 = 0$.

Since P_k contains F_1 if $k > 0$, the half- space in question contains points with $a \cdot (x - c) = 0$ and $a_k \cdot (x - c) < 0$. It follows that the half- space is given by the inequality

1) $(t_\ell a_\ell - t_k a_k) \cdot (x - c) \geq 0$.

We have proved that, in the case of all the bodies surrounding a fixed vertex c of P_0, we can find $t_1, \ldots, t_n$ such that the inequalities 1) define $H_{k,\ell}$. In general, starting with F_ℓ we define the numbers t_k by taking a chain of contiguous bodies from F_ℓ to F_k. Since the space is simply connected and 1) for $H_{k,\ell}$ and $H_{\ell,m}$ implies 1) for $H_{k,m}$, the numbers t_k are well-defined. Now, the lemma follows by replacing a_i by $t_i a_i$. q.e.d.

Note that in the general case of the above lemma c is any common vertex of P_k and P_1.

<u>Lemma 2</u> If the number of faces of P_0 is $2s$ and if we number the adjacent bodies P_i $(1 \leq i \leq 2s)$ in such a way that F_i and F_{i+s} are parallel, then the following relation holds $a_i = -a_{i+s}$.

proof) Assume that $a_i = -\alpha a_{i+s}$. And suppose that F_i and F_ℓ meet so that P_i and P_ℓ have the common face $F_m + u$. Then, $P_i = P_0 + u$ and P_{i+s} and $P_{\ell+s}$ have the common face $-(F_m + u) = F_{m+s} - u$.

By Lemma 1 $a_\ell - a_i$ and $a_{\ell+s} - a_{i+s}$ are orthogonal to this face. Hence, since $a_i = -\alpha a_{i+s}$ $\alpha > 0$, we must have $a_{\ell+s} = -\alpha a_\ell$. Similarly, $a_{k+s} = -\alpha a_k$ for all k. Therefore $a_i = \alpha^2 a_i$ which means that $\alpha = 1$. q.e.d.

We now define a single-valued function Φ on the lattice such that

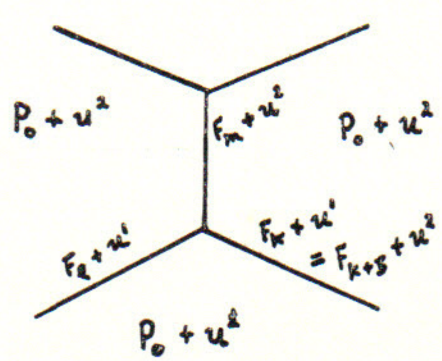
$$\Phi(0) = 0 \qquad \Phi(u + u_k) = \Phi(u) + a_k \cdot b$$

where b is any point in the common face of $P_o + u$ and $P_o + u + u_k$.

To show that the above function is single-valued we take any three bodies $P_o + u^1$, $P_o + u^2$, $P_o + u^3$ with a common vertex c. Let $u^2 = u^1 + u_k$, $u^3 = u^1 + u_\ell$, $u^3 = u^2 + u_m$, then the uniqueness requires that

$$\Phi(u^3) = \Phi(u^1) + a_\ell \cdot c = \Phi(u^1) + a_k \cdot c + a_m \cdot c.$$

If we show that $a_\ell - a_k = a_m$ in this general situation, the simple connectedness of R^n allows us to prove that $\Phi(u)$ is well-defined by shrinking a closed path through u and the origin to the point 0 at which $\Phi(0) = 0$.



Lemma 3 In this situation $a_\ell - a_k = a_m$

proof) From Lemma 1 we know that $a_\ell - a_k$ is perpendicular to $H_{k,l}$. A little reflection reveals that $F_m + u^2$ lies on the boundary of $H_{k,\ell} + u^1$. So,

$$a_\ell - a_k = \alpha a_m \quad \alpha > 0.$$

Similarly, $F_\ell + u^1$ belongs to the boundary of $H_{k+7,m} + u^2$. So,

$$a_m - a_{k+s} = \beta a \quad \beta > 0.$$

By Lemma 2 $a_{k+7} = -a_k$. This means that $\alpha = \beta = 1$ and $a_\ell - a_k = a_m$. q.e.d.

Furthermore, the uniqueness of the functional values of $\Phi(u)$ means that

$$\begin{aligned}\Phi(u_k + u_j) &= \Phi(u_k) + a_j\cdot(u_k + \tfrac{1}{2}u_j)\\ &= \tfrac{1}{2}a_k\cdot u_k + a_j\cdot u_k + \tfrac{1}{2}a_j\cdot u_j\\ &= \Phi(u_j) + a_k\cdot(u_j + \tfrac{1}{2}u_j)\\ &= \tfrac{1}{2}a_j\cdot u_j + a_k\cdot u_j + \tfrac{1}{2}a_k\cdot u_k.\end{aligned}$$

These equations imply that $a_k\cdot u_j = a_j\cdot u_k$.

We choose n of the vectors u_i which generate the lattice. For an arbitrary lattice point $u = \sum_{i=1}^{n} x_i u_i$, we construct the vector $a = \sum_{i=1}^{n} x_i a_i$. Then, from the relation $(a + a_k)\cdot(u + u_k) - a\cdot u = a_k\cdot u_k + a\cdot u_k + a_k\cdot u = 2a_k\cdot(u + \tfrac{1}{2}u_k)$, we obtain the equation $\Phi(u) = \tfrac{1}{2}\, a\cdot u$.

We now claim that the quadratic form $\varphi(x) = 2\Phi(u) = \sum s_{ij}x_ix_j$ with $s_{ij} = a_i \cdot u_j + u_i \cdot a_j$ defines the given parallelohedron P_0 and is positive definite.

In fact, $\varphi(x - u - u_k) - \varphi(x - u) = 2a_k \cdot (x - u - \frac{1}{2}u_k)$, since $u + \frac{1}{2}u_k$ is in $F_k + u$. We know that $a_k \cdot (x - u - \frac{1}{2}u_k) \geq 0$ if x is on the side of F_k containing $P_0 + u$. So, constructing a chain, we see that $\varphi(x - u) \leq \varphi(x - v)$ if x is in $P_0 + u$. Equality holds if and only if $P_0 + u$ and $P_0 + v$ are contiguous and x is on their common boundary. Thus, P_0 is determined by the equation

$$\varphi(x) \leq \varphi(x - v) \quad \text{for all v in the lattice.}$$

Finally, the form is positive definite. It is positive semi-definite because $\varphi(u) \geq 0$ for all u in the lattice by the way the form was constructed. This property extends to all real vectors by the homogenuity of the form. If $\varphi(x) = 0$ for some $x = 0$, then P_0 is unbounded which is contrary to the assumption at the start of this section.

This proves the theorem in its entirety.

§4 The Regulators and Characteristics which Correspond to the Sides of P_o [11][16]

We define a simplicial decomposition of the ambient space of P_o as follows. The vertices of the decomposition are the lattice points. The lattice points $u_{io}, \dots, u_{ik}$ are said to form a k-simplex if and only if $P_o + u_{io}, \dots, P_o + u_{ik}$ have a non-empty intersection. Let c be a vertex of $P_o + u_o$ and let L be the n-simplex determined by c. That is, L is the simplex $(u_o, u_1, \dots, u_n)$ such that c is the common vertex of $P_o + u_j$ $(0 \leq j \leq n)$ (These $u_o, \dots, u_n$ are different from the ones in § 3).

Definition A positive definite quadratic form $\psi(x) = \sum b_{ij}x_ix_j$ defines a set of primitive parallelohedra of the same type as those associated with P_o if the corresponding simplicial decompositions are the same. The form is also said to be of the same type as the form associated with P_o.

Thus, the angles of the sides of the cell around the origin may be varied without changing the type if the vertices of the corresponding simplices remain unchanged. This just means that the lattice is the same.

Let A be the matrix of the form in the previous section. Then, according to § 3, the function

$$F(c,x) = \tfrac{1}{2}\,\varphi(x) - Ax\cdot c = -\tfrac{1}{2}(\varphi(c) - \varphi(c-x))$$

satisfies the equation $F(c,u) = \alpha$ if $u = u_o, \ldots, u_n$ and the inequality $F(c,u) > \alpha$ otherwise. In fact, the lattice is determined by these conditions for all c. Let c_k denote the second common vertex of the parallelohedra $P_o + u_h$ with $h = k$. Let L^k be the simplex corresponding to c_k. Call the vertex of this simplex which is not among the u_i v_k. We have the relationships

$$F(c_k,u_h) - F(c,u_h) = \alpha_k - \alpha \text{ if } h \neq k \text{ and } > \alpha_k - \alpha \text{ if } h = k.$$

This means that the vector $A(c - c_k)$ is orthogonal to the common face of L and L^k. Moreover, this vector has positive projection on any of the vectors $u_k - u_h$ with $h \neq k$. We write $A(c - c_k) = \rho_k p_k$ with $\rho_k > 0$. In this equation p_k is a primitive integral vector; it is determined by k and the vectors $u_o, \ldots, u_n$. ρ_k is called the <u>regulator</u> of the edge (c,c_k) and p_k is called the <u>characteristic vector</u> of that edge.

It is convenient to define $F^*(c,x) = F(c,x) - \alpha$ and to write an arbitrary lattice vector as $u = \sum_{i=0}^{n} \theta_i u_i$ where $\sum_{i=0}^{n} \theta_i = 1$. Then, we have

$$\begin{aligned} F^*(c,u) &= \tfrac{1}{2}\Phi(u) - Au\cdot c - \sum_{i=0}^{n} \theta_i(\tfrac{1}{2}\Phi(u_i) - Au_i\cdot c) \\ &= \tfrac{1}{2}(\Phi(u) - \sum_{i=0}^{n} \theta_i \Phi(u_i)). \end{aligned}$$

Also, we have

$$\begin{aligned} F^*(c,v_k) &= Av_k\cdot(c_k - c) + F(c_k,v_k) - F(c,u_h) \\ &= Av_k\cdot(c_k - c) + F(c_k,u_h) - F(c,u_h) \\ &= A(c - c_k)\cdot(u_h - v_k) = \rho_k p_k\cdot(u_h - v_k) \end{aligned}$$

So, we have the formula

$$① \quad \rho_k p_k \cdot (u_h - v_k) = \tfrac{1}{2}(\Phi(v_k) - \sum_{i=0}^{n} \theta_i \Phi(u_i)),\ h \neq k$$
$$v_k = \sum \theta_i u_i \qquad \sum \theta_i = 1$$

which expresses ρ_k as a homogenous linear form in the s_{ij} for fixed $u_o, \dots, u_n$ and v_k. As was mentioned above ρ_k depends only on k and the vectors $u_o, \dots, u_n$. If we vary the coefficients s_{ij} of $\varphi(x)$ in a manner that keeps all the ρ_k positive, it would seem that the type of the form $\varphi(x)$ would remain the same.

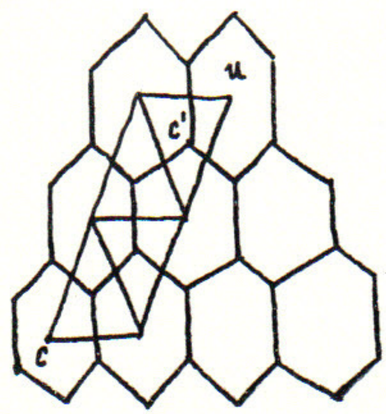
Theorem If we fix the lattice corresponding to φ and P_o, the coefficients s_{ij} may be varied in such a way that the ρ_k remain positive without changing the type of the form.

proof) We want to show that the lattice associated with the new form ψ is the same lattice associated with φ. The lattice associated with φ is determined by the conditions

$$F^*(c,x) = 0 \text{ if } x = u_o, \dots, u_n \quad \text{for all c.}$$
$$> 0 \text{ otherwise}$$

We will show that these conditions remain valid for the new ρ_k.

For that purpose see the figure on the next page. Let c' be a vertex of $P_o + u$ such that we can connect simplices $L, L^{(1)}_o, \dots, L^{(m)}$ satisfying the condition that u is always on the opposite side of the extension of the line segment $L^{(k)} \cap L^{(k+1)}$ than c is. This can be done by drawing a line from c to u and choosing simplices which meet that line segment.

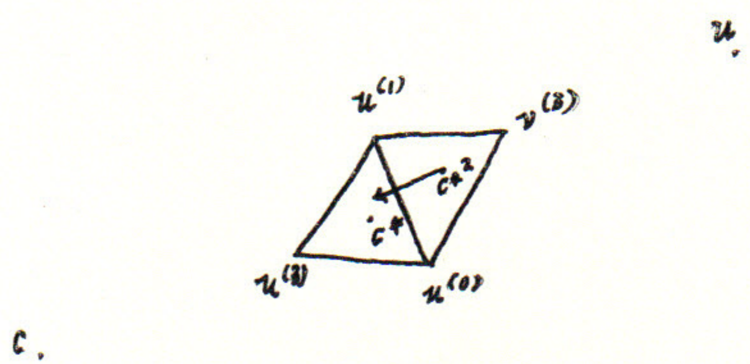


We have

$$F^*(c,x) - F^*(c_k,x) = -Ax\cdot(c - c_k) + Au_h\cdot(c - c_k)$$
$$= \rho_k p_k\cdot(u_h - x) \qquad h \neq k$$

So, in particular, one link in the chain of simplices gives the equation

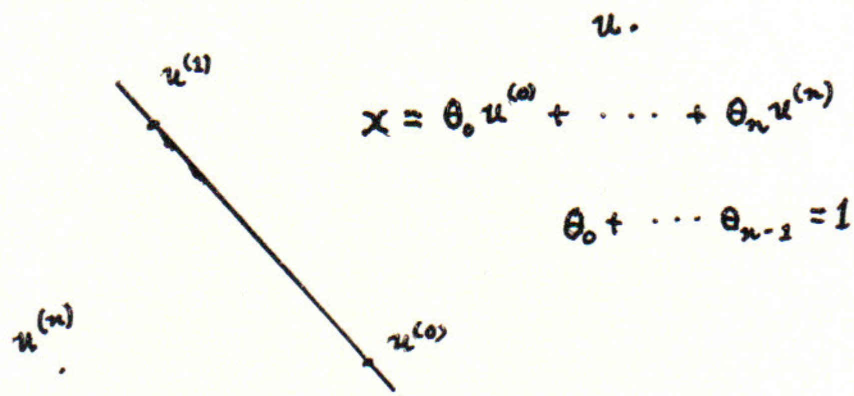
$$F^*(c^*,u) - F^*(c^{*2},u) = \rho_* p_* \, (u^{(i)} - u) \qquad i \neq n$$



where we have called the regulator associated with $[c^*, c^{*2}]$ ρ_*. Then, as mentioned previously,

$$p_* \,(u^{(i)} - u^{(j)}) = 0 \quad i \neq n \quad j \neq n$$
$$p_* \,(u^{(n)} - u^{(i)}) > 0 \quad i \neq n$$

We express $u = \sum_{i=0}^{n} \theta_i u^{(i)}$ with $\sum_{i=0}^{n} \theta_i = 1$



Since $u^{(n)}$ is on the opposite side of the hyperplane $x = \sum_{i=0}^{n-1} \theta_i u^{(i)}$ $\sum_{i=0}^{n-1} \theta_i = 1$ from u, we must have $\theta_n < 0$. This means that

$$\begin{aligned} p_* \,(u^{(i)} - u) &= p_* \,(u^{(i)} - \sum \theta_i u^{(j)}) \quad i \neq n \\ &= p_* \,(\sum_{i=0}^{n} \theta_i \,(u^{(i)} - u^{(j)}) \\ &= \theta_n p_* \,(u^{(i)} - u^{(n)}) > 0 \end{aligned}$$

Thus,

$$F^*(c^*, u) - F^*(c^{*2}, u) = \rho_* p_* \,(u^{(i)} - u) > 0$$

if ρ_* is.

Since $F^*(c,u) = 0$, we have by summation that $F^*(c,u) > 0$ if all the ρ_* are > 0. q.e.d.

§5 The Principal Domain [16]

Theorem Each residue class of lattice points modulo 2 contains one pair of points $\pm u_k$ such that P_o and $P_o + u_k$ are contiguous. In fact, the u_k are the points characterized by the conditions

$$*)\quad \varphi(u_k - 2v) \geq \varphi(u_k) \quad \text{for all } v \text{ in the lattice}$$

proof) If P_o and $P_o + u_k$ are contiguous, and if x is in F_k, then

$$\varphi(x) - \varphi(x - v) = 2Ax\cdot v - Av\cdot v = 2Av\cdot(x - \tfrac{1}{2}v) \leq 0 \quad \text{for all } v \text{ in the lattice}$$

and also $2Av\cdot(u_k - x - \frac{1}{2}v) \leq 0$ because $u_k - x$ is in P_o.

Adding these two inequalities, we get

$$Au_k\cdot u_k \leq A(u_k - 2v)\cdot(u_k - 2v) \text{ which is } *).$$

Conversely, if $\varphi(u_k - 2v) \geq \varphi(u_k)$ for all v in the lattice, then the point $x = \frac{1}{2}u_k$ satisfies

$$\varphi(x) \leq \varphi(x - v) \quad \text{for all } v \text{ in the lattice.}$$

Hence, x is in P_o, P_o and $P_o + 2x = P_o + u_k$ have boundary points in common, and therefore by the definition of a primitive parallelohedron they must be contiguous. q.e.d.

Now we examine the "principal domain", that is the domain of all forms of the same type as the "principal form"

$$\varphi(x) = 3x_1^2 + 3x_2^2 + 3x_3^2 - 2x_1x_2 - 2x_1x_3 - 2x_2x_3$$
$$= \sum_{i=1}^{3} x_i^2 + \sum_{i<j} (x_i - x_j)^2 .$$

In this case the u_k consist of the $2(2^3 - 1) = 14$ vectors whose coordinates are made up of exactly i, $i = 1,2,3$ positive or negative 1's and $3 - i$ zeroes.

If we transform the variables by the matrix A^{-1}, the equations of the half-spaces defining the extremal body

$$\varphi(x - u_k) \geq \varphi(x) \qquad \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = A^{-1} \begin{pmatrix} x_1' \\ x_2' \\ x_3' \end{pmatrix}$$

become

$$\varphi(u_k) + 2u_k \cdot x' \geq 0$$

$$3 \pm 2x_i' \geq 0 \qquad i = 1,2,3$$
$$4 \pm 2(x_i' + x_j') \geq 0 \qquad i < j$$
$$3 \pm 2(x_1' + x_2' + x_3') \geq 0$$

Changing the variables again to new coordinates, $y_1 = -x_1' \quad y_2 = -x_2' \quad y_3 = -x_3' \quad y_o = x_1' + x_2' + x_3'$, the equations become

$$3 + 2y_i \geq 0 \qquad i = 0,1,2,3$$
$$4 + 2(y_i + y_j) \geq 0 \qquad i < j$$
$$3 + 2(y_i + y_j + y_k) \geq 0 \qquad i < j < k.$$

We will now calculate the characteristics and regulators that correspond to this extremal body.

<u>Proposition</u> The coordinates of the vertices of the extremal body satisfy the equations

$3 + 2y_i = 0 \quad 4 + 2(y_i + y_j) = 0 \quad 3 + 2(y_i + y_j + y_k) = 0$
where (i,j,k,m) is a permutation of $(1,2,3,0)$.

proof) For $(1,2,3,0)$ we have the solution $x'_1 = 3/2$, $x'_2 = \frac{1}{2}$, $x'_3 = -\frac{1}{2}$. We check that $\varphi(x') + 2\sum y_i x'_i =$

$$\sum_{i=1}^{3} (x_i'^2 + x'_i) + \sum_{i<j} (x'_i - x'_j)^2 + (x'_i - x'_j) \text{ is } \geq 0$$
for all x' in the lattice.

The coordinates of the other 24 vertices are found by permuting the y_i-coordinates. q.e.d.

Given two vertices (i,j,k,m) and (j,i,k,m), the corresponding simplices have a common face. The vertices of this face are the vectors $u_i + u_j$, $u_i + u_j + u_k$, $u_i + u_j + u_k + u_m$, the last of which is zero. From the previous section we know that the corresponding characteristic has constant projection on these vectors. Thus, $p_i + p_j = 0$ and $p_i + p_j + p_k = 0$, if $i \neq 0$ and $j \neq 0$ (we have altered the previous notation, letting p_i stand for the i^{th} coordinate of the characteristic p). From these equations we find that $p_i = -1$ and $p_j = 1$. If $i = 0$, then $p_1 + p_2 + p_3 = p_j$ and we find that $p_j = -1$ with the other coordinates zero.

Now we use formula (1) from § 4 to calculate the regulators of the edge corresponding to the two vertices chosen above. Let $v_k = u_j$ and $u_h =$

a vertex of the face in the formula. Representing $u_j = \theta_0 u_i + \theta_1(u_i + u_j) + \theta_2(u_i + u_j + u_k) + \theta_3(u_i + u_j + u_k + u_m)$ $\sum_{i=1}^{3} \theta_i = 1$, we find that $\theta_0 = -1$, $\theta_1 = 1$, $\theta_2 = 0$, $\theta_3 = 1$. Then,

$$2\rho_{ij}(0 - (-1)) = \Phi(u_j) + \Phi(u_i) - \Phi(u_i + u_j)$$
$$= -2u_i \cdot u_j.$$

Thus, $i \neq 0$ $j \neq 0 \Rightarrow \rho_{ij} = -s_{ij}$. If $j = 0$, then $\rho_{ij} = \sum_{l=1}^{3} s_{li}$.

This shows that these regulators are the same as the ρ_k defined in section 1. Since in both cases the reduced domains are defined by the conditions $\rho_k > 0$, the two methods of reduction give the same reduced forms. We have only shown this for the case $n = 3$ and in fact it is not true for higher dimensions.

<u>Proposition</u> In the case $n = 3$ Voronoi's two methods of reduction give the same reduced forms.

§6 The Five Three Dimensional Parallelohedrons

From what we said in § 1 all positive quadratic forms in three variables are equivalent to a form of the following type

$$\Phi = \lambda x^2 + \lambda' y^2 + \lambda'' z^2 + \mu(y-z)^2 + \mu'(z-x)^2 + \mu''(x-y)^2$$

where all the coefficients are > 0.

From this general form we have 14 half-spaces which define the corresponding primitive parallelohedron. These are obtained by the technique of the previous section. There are 7 non-zero residue classes mod 2 represented by vectors u_k

$$u_k = (0,0,1),(0,1,0),(1,0,0),(1,1,0),(1,0,1),(0,1,1),(1,1,1)$$

such that the 14 surrounding points correspond to $\pm u_k$ $(1 \leq k \leq 7)$. By a theorem of Minkowski [11] there are at most 14 contiguous bodies surrounding a parallelohedron in three dimensions. By substitution in

$$\Phi(u_k) \pm 2 \left(u_k \cdot x\right) \geq 0 \quad .$$

we have the half-spaces given by the equations

$$-\tfrac{1}{2}(\lambda+\mu'+\mu'') \leq x \leq \tfrac{1}{2}(\lambda+\mu'+\mu'')$$
$$-\tfrac{1}{2}(\lambda'+\mu''+\mu) \leq y \leq \tfrac{1}{2}(\lambda'+\mu''+\mu)$$
$$-\tfrac{1}{2}(\lambda''+\mu+\mu') \leq z \leq \tfrac{1}{2}(\lambda''+\mu+\mu')$$
$$-\tfrac{1}{2}(\lambda'+\lambda''+\mu'+\mu'') \leq y+z \leq \tfrac{1}{2}(\lambda'+\lambda''+\mu'+\mu'')$$
$$-\tfrac{1}{2}(\lambda''+\lambda+\mu''+\mu) \leq z+x \leq \tfrac{1}{2}(\lambda''+\lambda+\mu''+\mu)$$
$$-\tfrac{1}{2}(\lambda+\lambda'+\mu+\mu') \leq x+y \leq \tfrac{1}{2}(\lambda+\lambda'+\mu+\mu')$$
$$-\tfrac{1}{2}(\lambda+\lambda'+\lambda'') \leq x+y+z \leq \tfrac{1}{2}(\lambda+\lambda'+\lambda'')$$

These inequalities define a truncated octahedron. Since the domain of reduced forms is the same as the domain of forms of this type there is only one type of primitive parallelohedron in three dimensions.

The imprimitive paralelohedrons are gotten by setting some of the normal coordinates $\lambda, \lambda', \lambda'', \mu, \mu', \mu''$ of the form equal to zero.

If we set $\mu = 0$ and $\mu' = 0$ and $\mu'' = 0$, the inequalities

$$-\tfrac{1}{2}\lambda \leq x \leq \tfrac{1}{2}\lambda$$
$$-\tfrac{1}{2}\lambda' \leq y \leq \tfrac{1}{2}\lambda'$$
$$-\tfrac{1}{2}\lambda'' \leq z \leq \tfrac{1}{2}\lambda''$$

define a cube.

If we set $\mu'',\mu' = 0$ or $\mu,\mu'' = 0$ or $\mu,\mu' = 0$, the inequalities

$$-\tfrac{1}{2}\lambda \leq x \leq \tfrac{1}{2}\lambda$$
$$-\tfrac{1}{2}(\lambda'+\mu) \leq y \leq \tfrac{1}{2}(\lambda'+\mu)$$
$$-\tfrac{1}{2}(\lambda''+\mu) \leq z \leq \tfrac{1}{2}(\lambda''+\mu)$$
$$-\tfrac{1}{2}(\lambda'+\lambda'') \leq y+z \leq \tfrac{1}{2}(\lambda'+\lambda'')$$

define a hexagonal prism.

If we set $\lambda'',\mu'' = 0$ or $\mu,\lambda = 0$ or $\mu',\lambda' = 0$ the inequalities

$$-\tfrac{1}{2}(\lambda+\mu') \leq x \leq \tfrac{1}{2}(\lambda+\mu').$$
$$-\tfrac{1}{2}(\lambda'+\mu) \leq y \leq \tfrac{1}{2}(\lambda'+\mu)$$
$$-\tfrac{1}{2}(\mu+\mu') \leq z \leq \tfrac{1}{2}(\mu+\mu')$$
$$-\tfrac{1}{2}(\lambda'+\mu') \leq y+z \leq \tfrac{1}{2}(\lambda'+\mu')$$
$$-\tfrac{1}{2}(\lambda+\mu) \leq z+x \leq \tfrac{1}{2}(\lambda+\mu)$$
$$-\tfrac{1}{2}(\lambda+\lambda') \leq x+y+z \leq \tfrac{1}{2}(\lambda+\lambda')$$

define the rhombic dodecahedron.
If we set $\mu'' = 0$ or $\mu = 0$ or $\mu' = 0$, the inequalities

$$
\begin{aligned}
-\tfrac{1}{2}(\lambda+\mu') &\leq x \leq \tfrac{1}{2}(\lambda+\mu') \\
-\tfrac{1}{2}(\lambda'+\mu) &\leq y \leq \tfrac{1}{2}(\lambda'+\mu) \\
-\tfrac{1}{2}(\lambda''+\mu+\mu') &\leq z \leq \tfrac{1}{2}(\lambda''+\mu+\mu') \\
-\tfrac{1}{2}(\lambda'+\lambda''+\mu') &\leq y+z \leq \tfrac{1}{2}(\lambda'+\lambda''+\mu') \\
-\tfrac{1}{2}(\lambda''+\lambda+\mu) &\leq z+x \leq \tfrac{1}{2}(\lambda''+\lambda+\mu) \\
-\tfrac{1}{2}(\lambda+\lambda'+\lambda'') &\leq x+y+z \leq \tfrac{1}{2}(\lambda+\lambda'+\lambda'')
\end{aligned}
$$

define the elongated dodecahedron.
The other possibilities are

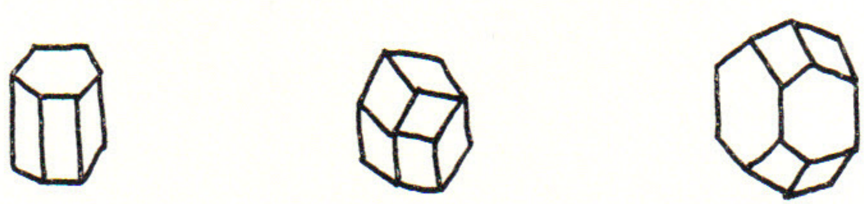
$\lambda = 0$ or $\lambda' = 0$ or $\lambda'' = 0$

or

$\lambda, \lambda' = 0$ or $\lambda', \lambda'' = 0$ or $\lambda, \lambda'' = 0$

which define the truncated octahedron

or $\lambda = \lambda' = \lambda'' = 0$ which defines a 2 dimensional hexagon.



hexagonal prism rhombic dodecahedron elongated dodecahedron

By Fedorov's result these are all the imprimitive parallelohedrons.[1]

§7 The Use of a Criterion of Voronoi to Calculate the Densest Lattice Sphere Packing

We now return to the problem of densest packing of spheres. As mentioned in the introduction, in terms of the associated quadratic forms, this is equivalent with maximizing $M/\sqrt[3]{D}$. Here M is the minimum of the form on the lattice points and D is its discriminant. There is a criterion for this related to Voronoi's first method of reduction. A positive definite form which is uniquely determined by the vectors that represent its minimum is said to be perfect. If there are s representations, then there are s conditions on the coefficients of the form. So, in any case, we must have $s \geq \frac{1}{2}n(n+1)$. Where n is the number of variables. This follows from the linearity of the conditions.

We define the adjoint form of a given positive quadratic form to be that form whose matrix is the transposed inverse of the matrix of the given form. Then, Voronoi's criterion can be stated as

Theorem (Voronoi[16]) A positive definite form corresponds to a maximum value of $M/\sqrt[3]{D}$ if and only if it is perfect and its adjoint form occurs in its reduced domain. The second condition is equivalent to saying that the adjoint form is representable as $\sum_{k=1}^{n} \rho_k (u_k \cdot x)^2$ in terms of the minimum vectors u_k of the given form.

Now we examine the forms corresponding to the sphere packings associated with the five different types of parallelohedra.

truncated octahedron: $x^2 + y^2 + z^2 + (y - z)^2 + (y - x)^2 + (z - x)^2$

The vectors representing its minimum are

$\pm(1,0,0)$ $(0,1,0)$ $(0,0,1)$ $(1,1,1)$, $(1,1,0)$, $(1,0,1)$, $(0,1,1)$

hence it is perfect.

cube: $x^2 + y^2 + z^2$

The vectors representing its minimum are

$\pm(1,0,0)$ $(0,1,0)$ $(0,0,1)$, hence it is not perfect.

hexagonal prism: $x^2 + y^2 + z^2 + (y - z)^2$

There is only two minimum vectors $\pm$ (1,0,0), so the form is not perfect.

elongated dodecahedron: $x^2 + y^2 + z^2 + (z - x)^2 + (y - z)^2$

The vectors representing its minimum are $\pm$ (1,0,0) (0,1,0), hence it is not perfect.

A_3 rhombic dodecahedron: $x^2 + y^2 + (y - z)^2 + (z - x)^2$

The vectors representing its minimum are $\pm$ (1,0,0) (0,1,0) (0,0,1) (0,1,1) (1,0,1) (1,1,1). In order to show that it is perfect it suffices to verify that the equations $\sum_{i,j} t_{ij} u_{ki} u_{kj} = 0$ where u_{ki} denote the above six vectors $k = 1,\ldots,6$, have no non-zero solutions. But, this can be easily verified by substitution. The matrix of the above form is

$$\begin{pmatrix} 2 & 0 & -1 \\ 0 & 2 & -1 \\ -1 & -1 & 2 \end{pmatrix}$$

and the matrix of the adjoint form is

$$\begin{pmatrix} 3 & 1 & 2 \\ 1 & 3 & 1 \\ 2 & 1 & 4 \end{pmatrix}.$$

The form corresponding to this last matrix can be represented in terms of the minimum vectors of the original form as

$$2y^2 + z^2 + 2(x + z)^2 + (x + y + z)^2 .$$

Thus, by Voronoi's criterion the form corresponding to the rhombic dodecahedron realizes the maximum possible value of $M/\sqrt[3]{D}$. This gives another proof of Minkowski's result that no other form not equivalent to the rhombic dodecahedron's form gives as dense a sphere packing.

§8 The Geometric Connection of the Voronoi Reduced Forms with the Realm of Minkowski Reduced Forms

The condition for the matrix Y corresponding to a form to be Voronoi reduced can be written

$$(A_o, {}^tAYA) - (A_o, Y) \geq 0 \quad \text{for all } A \text{ in } SL(Z)$$

We will write $Y = {}^tBB$ and $B = (B_1 \ldots B_n)$. The columns B_i of the matrix B can be considered as basis vectors of a lattice associated with the form. As A runs through all the integral unimodular matrices, the columns of BA run through all integral bases of the above lattice. Using the identity $\sum_{i \leq j} Y_{ij} = \sum_{i \leq j} B_i \cdot B_j$, we will say that a basis for a given lattice which minimizes the sum $\sum_{i \leq j} B_i \cdot B_j$ is a <u>reduced basis in the sense of Voronoi</u>.

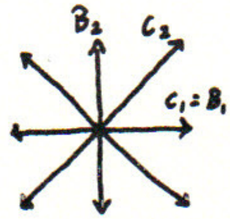
Minkowski defined a method of reduction which can be expressed in similiar terms. The conditions for the basis corresponding to a form to be <u>reduced in the sense of Minkowski</u>:

1) the C_i's are n vectors of succesively smallest length which determine an integral basis for the lattice (an equivalent formulation is that ${}^txYx \geq |C_i|^2$ whenever $(x_i, \ldots, x_n) = 1$ greatest common divisor).

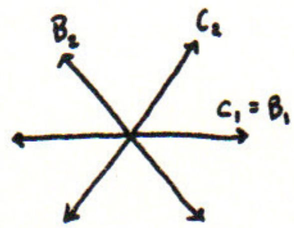
2) the angle between C_i and C_{i+1} is between 0 and $\pi/2$.

Since the successive minima of the lattice occur in pairs, it is clear that condition 2) can always be satisfied.

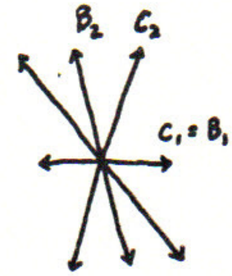
In the case n = 2 there are three types of lattices in this context.



rectangular lattice



hexagonal lattice



general lattice

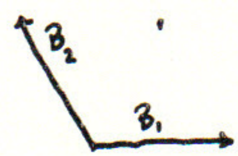
If we take C_1 and C_2 to be Minkowski reduced vectors in one of these types of lattices, then $B_1 = C_1$ and $B_2 = -C_1 + C_2$ will make the lattice Voronoi reduced. This can be proved by interpreting the following

<u>Theorem</u> The geometric conditions for a two dimensional form to be reduced in the sense of Voronoi are

1) $B_1 \cdot B_2 \leq 0$ 2) $-B_1 B_2 \leq B_2^2$ 3) $-B_1 B_2 \leq B_1^2$.

proof) That these conditions are necessary and sufficient follows from section 1 since they state that the $\rho_k \geq 0$. Another proof of their sufficiency can be given as follows

Let B_1, B_2 be a pair of vectors satisfying the above conditions.



We want to show that no other vector of the form $B_2 = mB_1 + nB_2$ gives a smaller value for the sum

$$(mB_1 + nB_2)^2 + B_1{}^2 + mB_1{}^2 + nB_1 \cdot B_2 =$$

$$n^2 B_2{}^2 + (2mn + n) B_1 \cdot B_2 + (m + m^2)\, B_1{}^2 + B_1{}^2$$

than the value $B_1{}^2 + B_2{}^2 + B_1 \cdot B_2$ that B_1 and B_2 give. As a preliminary step we note that if m,n minimize the above sum

1) $m = 0 \Rightarrow n = 1$
2) $n \neq 0$
3) if $m \neq 0$, $mn > 0$ since otherwise $2mn + n < 0$ and replacing n by $-n$ we would get a smaller sum.

Since $m^2 + m + n^2 = (m - n)^2 + 2mn + m \geq 2mn + n$, we want to minimize $(m - n)^2 + (m - n)$. Thus, we get $m - n = -1$ or $m = n$.

case 1) $m = n$

$$\text{sum} = \left[n^2 (B_1{}^2 + B_2{}^2) + 2n^2 B_1 B_2\right] + \left[n(B_1{}^2 + B_1 B_2)\right]$$

$\Rightarrow$ $m = n = -1$ since

$$B_1{}^2 + B_1 \cdot B_2 \geq 0 \qquad B_1{}^2 + B_2{}^2 + 2B_1 \cdot B_2 \geq 0.$$

case 2) $m = n - 1$

$$\text{sum} = n^2\left[(B_1{}^2 + B_2{}^2 + 2B_1 B_2)\right] - n\left[B_1{}^2 + B_1 B_2)\right]$$

$\Rightarrow$ $n = 1$ since

$$B_1{}^2 + B_2{}^2 + 2\, B_1 \cdot B_2 \geq 2(B_1{}^2 + B_1 \cdot B_2) \geq 0.$$

So the only two solutions are $B_2 = -B_1 - B_2$ and $B_2 = B_2$ which give the same value for the sum. q.e.d.