Concept as a Logical Relationship

- Here is another definition of a Concept to help us further elaborate this:
- ----- A concept is a logical relationship involving a predicative statement (subset of n times Cartesian product of the domain values and variables, instead of just a functional mapping). This logical relationship may also involve the question of the satisfaction of the concept (truth in terms of a specific knowledge representation). It may also involve the notion of a set of variable identifications in some model [data + algorithm][1]. And, it may also involve the notion of how a method for determining truth searches through the space of variable identifications inside of a pre-determined set of program search rules [logic + control][2] as a part of determining what the algorithm used will be.

2nd Order and 1st Order Logical Definitions

- 1st order logical definitions of sets of objects: They are of the form a set of objects = { x | x exists and satisfies a propositional logic functional predicate f(x)}
- 2nd order logical definitions of sets of objects: They are of defining first the form a relational function which to be applied to a of objects =
- Set of pairs of Function elements -> (x,y) such that x exists and f(x)=y.
- The pairs determine the relationship we are trying to characterize and also a function f(x) is determined from the pairs if the mapping is injective or one-to-one and surjective or onto the whole range of the 2nd part of the pair space. See the reference 10] for much more on this.
- 'If we define the Number One as the "set of all sets which have the set of no element in them", then this is a 2nd order predicative definition. A problem arises in this, as explained by Hilbert [7]. The way sets defined by 2nd order predicates are determined to be equal is by checking the functional values on all the terms of set elements in the basic universe of objects.

- Theoretically, there are an infinite number of possible objects, But, when we start out the calculation to check for equality we only have a finite set of data values. When Hilbert and Ackermann wrote their book on mathematical objects the techniques computer programming was just being thought out. They thought that since these considerations keep up from using formal logic to prove the existence of an infinity of numbers we couldn't compute things determined this way. It was not conceived (until Curry and Church thought it out later) that we could have dynamic memory allocation of things like what are called now "streams" in functional programming. Functional programming data streams applied to sets assume what we call in set theory as "The Axiom of Infinity" as justification for reasoning inductively and forward-chaining logically ahead for purposes of computation. The reason this works is basically the same reason that "Zeno's Paradox" does not keep us from defining the real continuum.

Frege's and Von Neuman's 1st Order Definitions

- Let the Number Zero be the set whose only element is the empty set:

Z= Zero = { x | x does not exist in any set}

One = { Z}, the Two is defined as {One} or {{Z}}

Alternatively, according to Von Neumann, for ordinals we can say Two = {Z, {Z}}

As mentioned above, Kant believed that numbers were "transcendental objects" not "mental objects" as Plato did. This definition combines and unifies the two ways of looking at numbers as well as introducing the machinery of Aristotelian logic and syllogisms further into Kant's ideas. Investigating the subtle but fascinating distinctions between these two equivalent ways of defining one (as an cardinal and ordinal) was to occupy the though of Turing at the same time he helped invent theoretically our modern day digital computer.

Frege's Definition of Numbers using 2nd Order Logical Predicates

- Frege believed that numbers were objects. He also believed that Plato that concepts were objects. However, he did not believe that numbers were concepts. He believe that numbers were values or extensions of concepts. He wrote his Begriffsschrift (concept notation) lectures in 1879 in which we laid out the logical foundations for his idea of a Humean and Kantian theory of Leibnitzian identity in logical propositions . He also explained in this paper how these ideas could be used to give a better idea of what a the concept of "mathematical function" is. And, later he wrote "Basic Laws of Arithmetic" [17] in 1893 in which he attempted to formalize his above idea of numbers being "extensions of concept". Then, however, Bertrand Russell after reading the Basic Laws came up his paradox related to how Frege defined his values or extensions of concepts. To this day many logical positivist philosophers believe this was a "knockout blow" to Frege's ideas. However, in his lectures from 1910 [16] Frege leaves out his Axiomx V and VI from the Grundgesetze which led to the Russell paradox problem from his theory of extensions. What is left is a clear and workable system of mathematical logic in which set theory, a theory of identity in statements of propositional logic, mathematical functions, ordinal numbers, cardinal numbers can be defined.

Hilbert's 2nd Order Definitions of Zero and Equality with Class Predicates

- The mathematician David Hilbert in his book [7] makes the definition Zero = 0(F): ~(E x)F(x) as an "operator" and not a "set" which is a differrent defintion than Frege's

(which means verbally, there is no x for which F is true)

We can create a correspondence between "operators" F and "sets" S by associating the set of all elements s in S which make operators f in F true. Then, if we translate this definition into terms of sets, what it says is that Zero (as a set) = the set with no elements in it, what we have called Z above (so it is the empty set and this definition given here is same way Frege defined Zero). The reason is that if any element was in Zero, then applying the identity function to it under the above correspondence would contradict this.

- Definition: ==(x,y) : (F)(F(x)=>F(y)&&F(y)=>F(x))
- That is, he calls x identical with y, if any predicate which holds for x also holds for y and vice versia. [10]

Hilbert's 2nd Order Definition of One as an Operator (a property of sets)

- The next definition of a number in Hilbert's book is a little harder to understand. Definition The Number One = 1(F): (Ex)[F(x) & (y) (F(y) implies ==(x,y)] as an "operator" [7].
- Verbally, this says, "There is an x for which F(x) holds, and any y for which F(y) holds is identical with this x." If we apply the correspondence between "operators" F and "sets" S of the previous slide we can see what set this corresponds to. It corresponds to a set S of elements x (which we are calling a set of elements "One") such that if the term exists and it is true that F(x) is true (can be verified for the propositional function F (x)) and it is also true that there is a term y such that F(y) is true, then x==y. Or, in other words, we define set as being determined by its elements. Then, we say, the "number one" is the set determined by only one element.
- So, this is a new way to do the definition is that is different than defining "One" = {{Z}} or "One" = {Z {Z}} and it uses functions as well as sets or "objects".
- Thus, this definition utilizes the idea of defining the number one as a Functor (see previous slide for definition of this). It defines the number as a "property of sets" instead of a "property of predicates". In Frege's book this is stated as, "The number one belongs to a concept F, if the proposition that a does not fall under F is not true universally, whatever a may be, and if from the propositions "a falls under F" and "b falls under F" if follows universally that a and b are the same."[1]

2nd Order Definition of "One" as a set (a property of predicates)

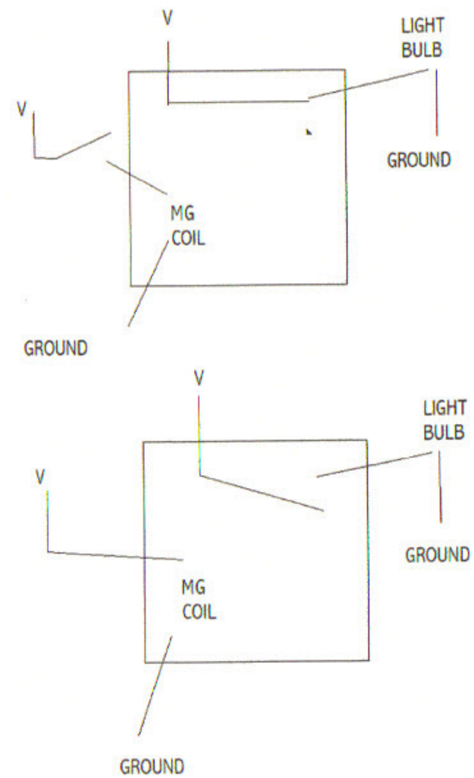
- As we said above the definition of "One" as the "set of all sets which are equivalent to the set with only the empty set as a member" is a 2nd order definition. Several things must be in our set theory in order to allow this definition. We must have an "axiom of infinity" , an "axiom of the universal set" V, and an ability to evaluate logical predicates functionally and allocate new terms dynamically. Here **we need to apply the correspondence between "equivalence classes of sets", ~(S) and "sets" S. The way 2nd order predicates come in here is in the "equivalence class relations" that are defined by the operator definition in the previous slide. In order to specify "sets which are equivalent to the set with only the empty set as a member" we must have a universal set V in which these sets occur and are defined by logical predicates with respect to. In addition, we must be able to pick out elements from all sets and map equivalences between two different occurances of the "set with only the empty set as a member" in the universe V of all sets. This requires what is called the "Axiom of Singletons" to be true. It says that we assume that for every object x, the set {x} = {y such that y==x} exists. Here the equality of sets is defined element-wise (x==y if for all z existing in x, z exists in y and vice-versia . It is not defined as above where x == y, if and only if any predicate which holds for x also holds for y and vice versia. Given any "well-ordered" set it has a least element and this allows us check whether it is the only element in "one", the empty set in this model.**

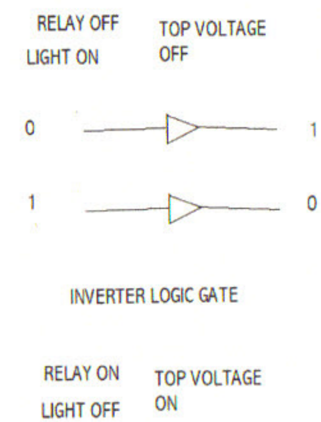
Quine's New Foundations for Set Theory

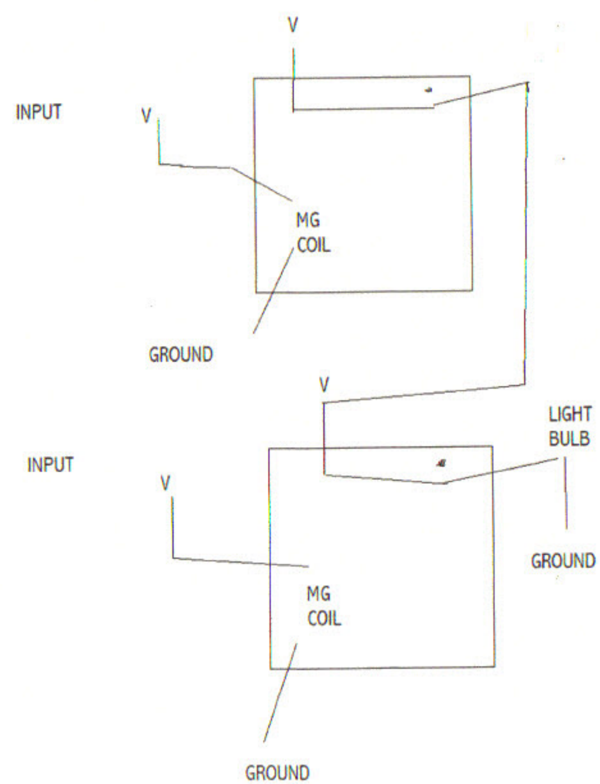
- **There is at least a third possibility, other than Frege's, Von Neumann's and Hilbert's, for defining the "Number One" In the 1940s and 1950s, the student of Bertrand Russell, Quine was a professor of the philosophy of mathematics and logic at Harvard. He published as theory of sets in which we can assume that the "set of all sets" necessarily exist as an axiom at the start. This universal set V is/was the same set as Plato and Parmenides discussed in their dialogue and called the set of "Others". But, he does not define "One" the same way Plato and Parmenides did. He defines it as meaning V, what Plato and Parmenides called "the Others". In this theory using several other more standard set theory axioms from the Zermelo-Frankel set theory it is possible to prove what we call the "Axiom of Choice" as a theorem of the theory and not an axiom. If you add another axiom, "The axiom of Singletons" this can happen. It says that we assume that for every object x, the set {x} = {y such that y==x} exists. The number Z "Zero" (or empty set) is defined as before. But, One, has been defined here as the set whose only element is V (the whole Universe of sets). The rest of the natural numbers can also be defined in this form of set theory, but, as you would think their definitions will have very different meanings.**

USING A MECANICAL DEVICE TO DEFINE THE NUMER ONE

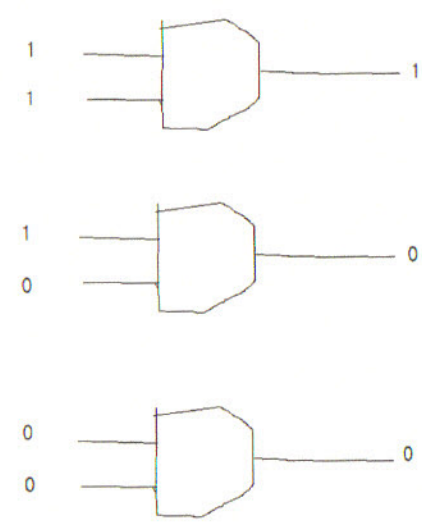
TWO DOUBLE THROW
RELAY LOGIC GATES

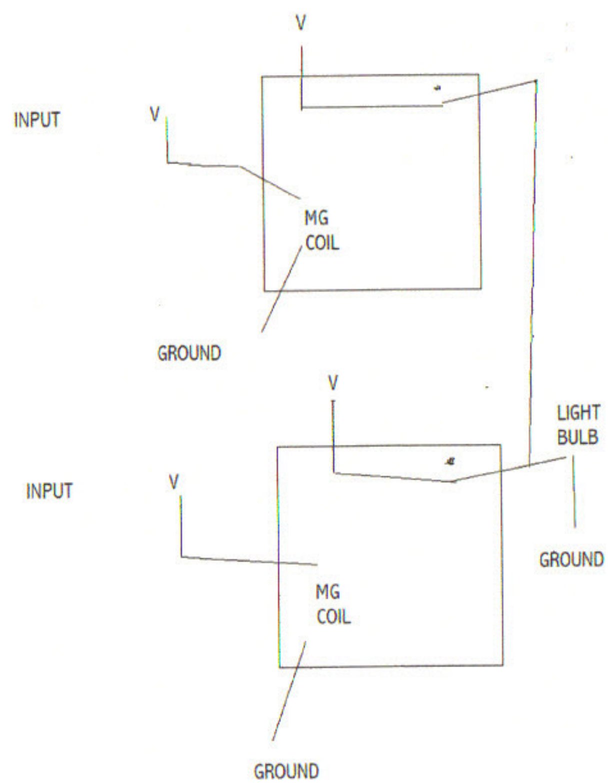




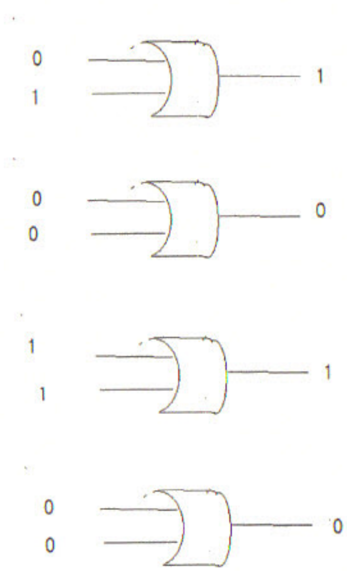


NAND LOGIC GATE

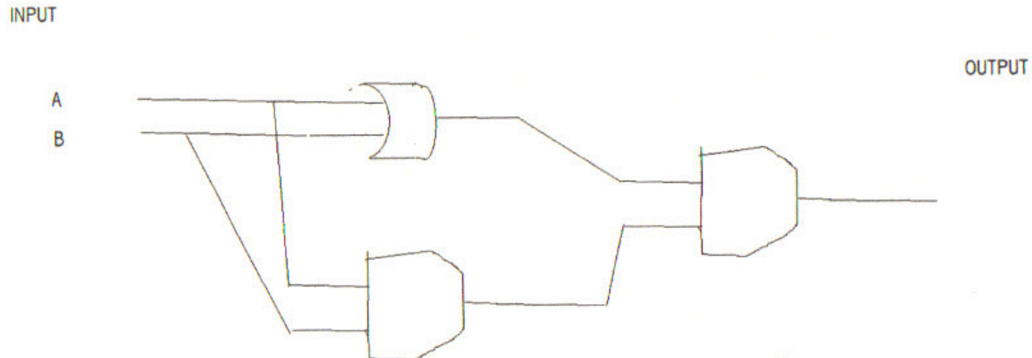




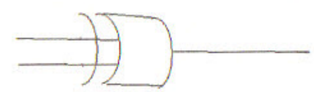
NOR LOGIC GATE



CONSTRUCTION OF THE EXCLUSIVE OR LOGIC
GATE OUT OF TWO NANDS AND AN OR



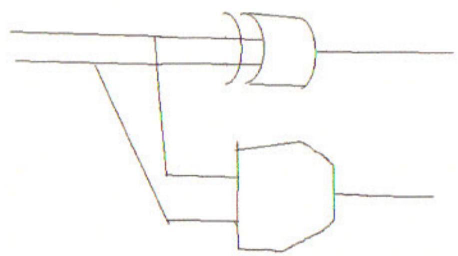
XOR GATE

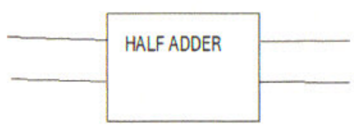


B INPUT

A INPUT

XOR	0	1
0	0	1
1	1	0

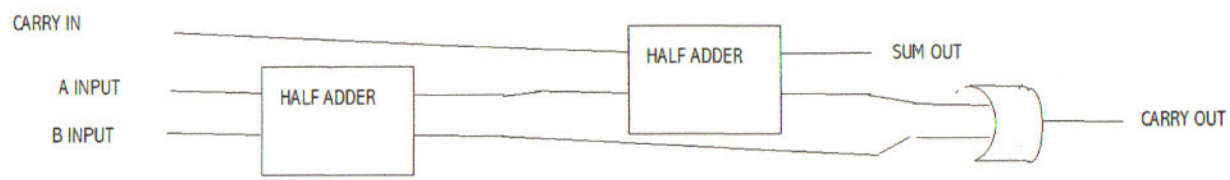


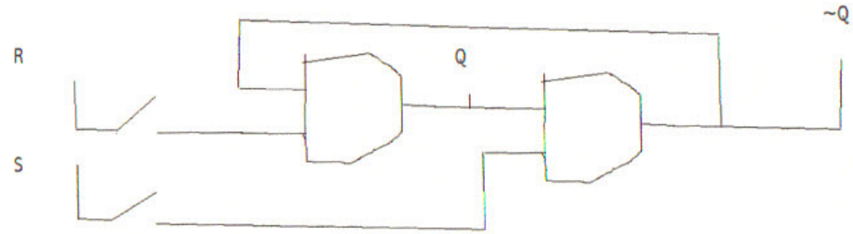


SUM	0	1
0	0	1
1	1	0

CARRY	0	1
0	0	0
1	0	1

A FULL ADDER CAN BE CONSTRUCTED FROM TWO HALF ADDERS AND AN OR GATE





INPUTS	OUTPUTS
S R	Q ~Q
! 0	1 0
0 1	0 1
0 0	Q ~Q
1 1	not allowed

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